Instruments for intergenerational risk sharing

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November 2009

Abstract

I study the intergenerational sharing of macroeconomic risk in a many period, two overlapping generation model of a closed economy. The market solution leads to too little risk sharing with future generations, compared to the social optimum. Contrary to results obtained in a single-period model, it is not possible to emulate the optimum using pay-as-you-go and funded pensions. Funded pensions with an option to retain capital or stay underfunded are introduced as an option to increase risk sharing but cannot exactly replicate the social planner.

JEL codes: E21, H55, J18
Keywords: funded and unfunded pensions, fiscal policy, risk sharing, capital/labor ratio.

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1 Introduction

Macroeconomic shocks happen for a variety of reasons and show up in productivity and asset values. They cause fluctuations in welfare, and if the economy is inhabited subsequently by a number of generations, these generations would have higher expected welfare if they could insure with each other against these shocks. Insurance requires trading of contingent assets, which may not be possible if the generations are non-overlapping. Beetsma and Bovenberg (2006) show that a pension system with a wage-linked first pillar and a defined benefit (DB) second pillar can (in principle) help generations pool their risk to insure against shocks. The authors consider inflationary shocks, shocks in productivity and in the rate of depreciation. The pension system is shown to replicate the optimal solution of an economic planner if the sizes of the two pillars, and the provisions of the contract, are just right.

A drawback of Beetsma and Bovenberg (2006) is that their model only considers two generations, which overlap during one period. In other words, there are no generations which do not overlap. This makes it impossible to smooth out shocks over a longer time, which limits the beneficial effect of intergenerational insurance. It also sidesteps the problem of sharing risks between generations that are not alive at the same time. This paper aims to remedy this drawback by moving the analysis to a framework in which there are two generations present in any period, but infinitely many periods. In the text, results that deviate from those in Beetsma and Bovenberg (2006) are emphasized while those that are similar receive less attention. Ideally, we would show which pension institutions come closest to the command optimum, and indicate the approximate size of these institutions using numerical simulation. At the present stage, however, we can only present negative results: the optimum is not easily replicable and further study is needed to find the closest substitute.

Our main thesis is that the efficacy of pension institutions as means to insure shocks should be judged on two criteria: the ability to share and insure shocks and, probably more importantly, the characteristics of the steady state of the economy under these institutions. It turns out that, through their effect on the decision to accumulate capital, pension institutions may have a large impact on the steady state level of production in a closed economy. This impact works both through the "replacement" effect that arises when alternative assets and pseudo-assets such as pay-as-you-go are created and replace capital investment, and through effects on the riskiness of old-age income.

Our results include not only a theoretical discussion of the impact on different shocks on different generations, but also a number of numerical illustrations that give a sense of the size of these effects. These illustrations also show the size of the impact on the steady state by the distribution of shocks over the generations. The general principle is that, if the risk inherent in capital is shouldered by many generations instead of just one, the steady state level of capital is higher. These changes have important welfare effects.

We derive the command optimum in section 3 and show that in it, shocks are shared among many generations. The market equilibrium is derived in section 4. We compare the steady state and the distribution of shocks in a number of institutional settings: laissez faire, with pay-as-you-go and funded pensions.

Section 5 then goes into the question what an optimal system would look like. Section 6 concludes.

2 Related literature

The current work is close in spirit and execution to Bohn (1998, 2009). The main conclusion of the former is the inefficiency of safe government debt. With the risks already borne by the young generation in a *laissez faire* setting, the added responsibility for government income shocks is not something they particularly desire. An efficient system would put rather more risk on the old generation than seems to be the case in the current empirical setting. Bohn notices that this can be explained by assuming a degree of risk aversion that increases with age.

This analysis differs from that in Matsen and Thøgersen (2004), which has similar goals, by the nature of the underlying shocks. Where Matsen and Thøgersen (2004) use historical statistical data on stock market returns and wage growth, we model these financial outcomes as the result of underlying shocks in productivity and depreciation. As Thøgersen (2006) notes, modelling the deeper determinants of the returns on capital and labor is desirable in principle, but the correlations between different assets that theoretical models predict have a tendency to be higher than those in the real world.

Gollier (2007) also looks at the command optimum version of intergenerational risk sharing and compares it to real-world pension funds that are subject to discontinuity risk. This prevents them from running risky strategies that have a higher expected value. Gollier finds that this, second-best, pension system provides less than half the gain from intergenerational risk sharing.

Sinn (1995) makes the more general point that (social) insurance against bad outcomes increases the amount of risk that people are willing to take, possibly bringing it closer to the optimum. We make a similar point in this paper regarding the beneficial effect that the insuring of shocks in the return to capital has on capital buildup.

3 The command optimum

Each agent lives for two periods and consumes in each period; when they are young, agents maximize expected utility

$$\mathrm{E}_{t-1}\left[U(c_{y,t-1}, c_{o,t})\right] = u(c_{y,t-1}) + \beta \mathrm{E}_{t-1}\left[u(c_{o,t})\right].$$

This maximization is done through the saving decision: agents inelastically supply one unit of labor when young and nothing when they are old. There are n_t young agents in period t, and n_{t-1} old agents. Each agent thus lives into old age.

Production is equal to

$$Y_t = A_t F(K_t, n_t)$$

where total factor productivity A_t is stochastic. The stock of capital K_t is what is left after depreciation plus the savings of the previous period so that in the absence of international trade, the resource constraint is

$$[K_{t+1} - (1 - \zeta_t)K_t] + n_t \cdot c_{y,t} + n_{t-1} \cdot c_{o,t} = Y_t.$$

In each period, there are two shocks: total factor productivity equals

$$\log A_t = \log \bar{A} + \omega_{A,t}$$

where $\omega_{A,t}$ is normally distributed with mean zero and variance σ_A^2. For the sake of simplicity I abstain from modelling permanent shocks to productivity or, for that matter, autocorrelation. Depreciation of capital from the previous period is, similarly,

$$\log \zeta_t = \log \bar{\zeta} + \omega_{\zeta,t}$$

with $\omega_{\zeta,t} \sim \mathrm{N}(0, \sigma_\zeta^2)$. This means that agents are uncertain of two aspects of the capital in which they must invest at time t: its marginal productivity $A_{t+1} \cdot F_K$ and its rate of depreciation ζ_{t+1}. The path of population, n, is known. Stochastic variables are indexed by the time at which they are revealed: A_t is known at time t, but not before.

3.1 Intergenerational risk sharing in open and closed economies

Before we formally solve for the optimal policy, let us take a minute to reflect on what is known about optimal risk sharing in open and closed economies. Consider an economy as just described, in which the shocks are to be shared between generations. Intergenerational risk sharing between generations that are alive at the time of the shock are just transfers from one generation to the other. For these to take place, it does not matter whether the economy is open or closed.

If we want to share with future, unborn, generations as well, the situation changes. As these generations are not around to supply or receive transfers, any change in their income has to be effected throught today's saving decision. For example, take an economy that is currently at its steady state. When a negative shock hits and the present generations decide to share this shock with their future offspring, they can reduce their consumption with less than the amount of the shock, taking the difference out of their saving. For what happens next, it matters whether the economy is closed or open.

In a closed economy, or one that is large enough to influence the interest rate, the reduction in saving leads to a fall in the capital stock. This will decrease the capital/labor ratio to an inefficient level. Future generations can decide that they, too, would like to move the effects of the shock to their offspring and fail to bring K/L back to its efficient level. Doing so entails a cost, however, as their productivity is lower than it would have been in the steady state.

When the economy is small and open, the interest rate is independent of local saving and investment. This allows the economy to move to an efficient capital/labor ratio immediately and stay there, even while it is sharing shocks between generations. In the previous example, the generations affected by a negative shock could have propped up their consumption using a loan from the international capital markets. Following generations may or may not decide to pay off this loan, but their decision carries no cost in terms of lost productivity.

For the open economy, a well known result in the literature is that shocks should be smoothed over all future generations by negating their effects on the international capital market, paying (or receiving) only the interest while leaving the principal outstanding (see, for instance Hall 1978, Gordon and Varian 1988, Obstfeld and Rogoff 1996, section 1.3). This is an intuitive result: by not repaying the principal, *all* current and future generations get an equal share in the effects of the shock.

The above result is not valid, however, in a closed (or large) economy where the effects of a shock cannot be parked on the capital market without further efficiency costs. As noted, for each period away from the efficient capital-labor ratio the closed economy pays a penalty in terms of efficiency. The price for sharing with distant future generations may well be higher than the benefits, and so we should expect this economy to move back to the steady state in due time, leaving a limited number of generations to absorb the effects of the shock. We will now derive the solution of the (closed-economy) problem stated in the previous paragraph and see that it indeed has these properties.

3.2 Solution

The planner considers a social welfare function in which each generation's utility receives a positive weight, proportional to the generation's size. The planner discounts utility in future periods with the personal subjective discount factor β and discounts future generations with factor $\mu \leq 1$. Thus, the planner's objective function is $\mathrm{E}_t \Psi$, with

$$\Psi(\{c_{o,s}, c_{y,s}\}_{s=t,\infty}) = n_{t-1}u(c_{o,t}) + \sum_{s=t}^{\infty} \mu(\mu\beta)^{s-t} n_s \left(u(c_{y,s}) + \beta u(c_{o,s+1})\right)$$

Maximization gives the Euler conditions

$$u'(c_{o,s}) = \mu u'(c_{y,s}) \qquad \forall s \geq t. \tag{1}$$

and

$$u'(c_{y,s-1}) = \beta \mathrm{E}_{s-1}\left[u'(c_{o,s})\left(1 + r_s\right)\right] \qquad \forall s \geq t+1. \tag{2}$$

where

$$r_s = A_s F_K(K_s, n_s) - \zeta_s \tag{3}$$

These results are well known (see for instance Beetsma and Bovenberg 2009), see the appendix for a derivation.

The generational discount factor μ can play a useful role in analysis. Bohn (2009) shows that for any market solution there is a parameter μ such that the planner's problem with that parameter generates the same initial allocation. Bohn calls this the *comparable efficient allocation* and proceeds to compare the allocation of risk in the market solution to that in the planner's optimum using equilibrium properties that hold for any μ.

3.3 Log linearization

There are multiple ways to proceed with solving this model. One is to obtain the solution through numerical means, using perturbation methods around the steady state without uncertainty (see, for instance, Schmitt-Grohé and Uribe 2004). The other is to approximate using log-linearization. The latter allows a bit more insight into the workings of the model, and it is the approach I will use in this paper. I will however use the numerical solver Dynare++ (Kamenik 2007) to check some of the assumptions that go into log-linearization.

We start in a situation in which the economy at time $t-1$ happens to be at its steady state. This state is defined by the fact that stochastic variables at $t-1$ are equal to their expectations, population growth is zero and the capital stock has reached a level at which it would be constant if the shocks continued to be equal to their expectations.[1]

It is useful to write down two resource constraints. In the steady state[2] we have that $n_t = n$ for all t and $K_t = \bar{K}$. So,

$$n(\bar{c}_y + \bar{c}_o) + \bar{\zeta}\bar{K} = \bar{A}F(\bar{K}, n). \tag{4}$$

Outside of the steady state, the resource constraint becomes

$$n(c_{y,t} + c_{o,t}) + K_{t+1} = A_t F(K_t, n) + (1 - \zeta_t)K_t. \tag{5}$$

Shocks materialize on the right-hand side of this equation and have to be absorbed on the left-hand side. We will write this constraint in log-deviations from the steady state. Start by defining $T_t = A_t F(K_t, n) + (1 - \zeta_t)K_t$, the total available resources, and subtract and divide by $\bar{T}$:

$$\frac{n(c_{y,t} + c_{o,t}) + K_{t+1} - \bar{T}}{\bar{T}} = \frac{T_t - \bar{T}}{\bar{T}}.$$

Use a hat to denote log-deviation, so that $\hat{x} = \log(x) - \log(\bar{x})$. The approximation $x - \bar{x} \approx \hat{x}\bar{x}$ then holds so that the constraint (5) becomes

$$a_{cy}\hat{c}_{y,t} + a_{co}\hat{c}_{o,t} + a_{KS}\hat{K}_{t+1} = a_Y \omega_{A,t} - a_\zeta \omega_{\zeta,t} \tag{6}$$

with $a_{cy} = n\bar{c}_y/\bar{T}$, $a_{co} = n\bar{c}_o/\bar{T}$, $a_{KS} = \bar{K}/\bar{T}$, $a_Y = \bar{A}F(\bar{K}, n)/\bar{T}$ and $a_\zeta = \bar{\zeta}\bar{K}/\bar{T}$. Note again the exogenous shocks on the right-hand side and the endogenous response on the left, where $a_{cy} + a_{co} + a_{KS} = 1$.

To distribute the shocks among the three endogenous variables, we use the Euler conditions (1) and (2). Write the first of these in log-deviations as

$$\hat{c}_{y,s} = \frac{\sigma^o}{\sigma^y}\hat{c}_{o,s} \tag{7}$$

Here we use that $\hat{u}'(c) \approx \sigma\hat{c}$ with σ the coefficient of relative risk aversion at the central point $\bar{c}$ (see the appendix). From this and (1) we see that the consumption level of a young generation can be used to compute that of the overlapping old generation. The relationship between one generation's consumption while young and consumption while old follows from (2), which we can write as

$$u'(c_{y,s}) = \beta(1 + \bar{r})u'(\bar{c}_o)\mathrm{E}_{s-1}(\exp(\hat{r}_s - \sigma^o\hat{c}_{o,s}))$$

[1]...but without the public being aware of this; that is, the expected variance is not zero.

[2]A bar over a variable indicates that it is at its steady state value (K) or, in the case of stochastic variables, that they are equal to their expectations (A, ζ).

(this follows Beetsma and Bovenberg 2009). Here, $\hat{r}_s$ is the logarithmic deviation of $1+r_s$ from its expected value. By finding an expression for the expectation on the right-hand side in terms of parameters, we can solve this equation for $\bar{r}$ and thus for the steady state level of capital. Beetsma and Bovenberg (2009, Appendix B) show that

$$\hat{r}_s \approx a_K \omega_{A,s} - a_{\zeta K} \omega_{\zeta,s} \tag{8}$$

where $a_K = \bar{A}F_K(\bar{K},n)/(\bar{A}F_K(\bar{K},n)+1-\bar{\zeta})$ and $a_{\zeta K} = \zeta/(\bar{A}F_K(\bar{K},n)+1-\bar{\zeta})$.

We will write $\hat{c}_{o,s}$ in terms of the shocks using the resource constraint (6) above. For this, we conjecture that $\hat{K}_{s+1} = \theta_K \hat{c}_{y,s}$ for some θ_K and use this and (7) to write

$$((a_{cy} + \theta_K a_{KS})\frac{\sigma^o}{\sigma^y} + a_{co})\hat{c}_{o,s} = a_Y \omega_{A,s} - a_\zeta \omega_{\zeta,s}$$

so

$$\hat{c}_{o,s} = \left[\frac{\sigma^y}{(a_{cy} + \theta_K a_{KS})\sigma^o + a_{co}\sigma^y}\right][a_Y \omega_{A,s} - a_\zeta \omega_{\zeta,s}]$$

We will have a means to assess the validity of this conjecture when we solve this model using perturbation methods. This is done in section 3.4.

Now write $\mathrm{E}_{s-1}[\exp(\hat{r}_s - \sigma^o \hat{c}_{o,s})] \equiv \mathrm{E}_{s-1}[\exp(\phi_s)] = \exp(\mathrm{E}_{s-1}[\phi_s] + \mathrm{Var}[\phi_s]/2]$. With the above results, we write

$$\phi_s = \left[a_K - \frac{\sigma^o \sigma^y a_Y}{(a_{cy} + \theta_K a_{KS})\sigma^o + a_{co}\sigma^y}\right]\omega_{A,s} + \left[a_{\zeta K} + \frac{\sigma^o \sigma^y a_\zeta}{(a_{cy} + \theta_K a_{KS})\sigma^o + a_{co}\sigma^y}\right]\omega_{\zeta,s}. \tag{9}$$

This allows us to see that $\mathrm{E}_{s-1}(\phi_s) = 0$ and write the variance of ϕ_s as a function of σ_A^2 and σ_ζ^2. The Euler condition (2) for the median system then reads

$$u'(\bar{c}_y) = \beta(1 + \bar{r}(\bar{K}))u'(\bar{c}_o)\exp(\mathrm{Var}(\phi_s(\theta))/2). \tag{10}$$

As with the first Euler equation, it is useful to derive a version in log-deviations as well. Rewrite (2) as

$$\sigma^y \hat{c}_{y,s} = \mathrm{E}_s[\hat{r}_{s+1} + \sigma^o \hat{c}_{o,s+1}].$$

It is not correct to conclude from (8) above that the expectation of $\hat{r}_{s+1}$ is zero, as the change in r follows from different causes. The causes in (8) are unexpected shocks in productivity and depreciation. Since these shocks are transitory, the expectation for next period's value is zero. But in this case, *previous* shocks have lead to a change in the capital stock, which will cause a foreseeable change in its return. This is what is described by the current formula.

Remember that $\hat{r}_t$ is the log-deviation of $1 + r_t$. This means that

$$\sigma^y \hat{c}_{y,s} = \sigma^{dK} \hat{K}_{s+1} + \sigma^o \hat{c}_{o,s+1} \tag{11}$$

with

$$\sigma^{dK}(\bar{K}) = \frac{-\bar{K}\bar{A}F_{KK}(\bar{K}, \bar{n})}{1 + \bar{A}F_K(\bar{K}, \bar{n}) - \bar{\zeta}}. \tag{12}$$

If it happens that $\bar{\zeta} = 1$, σ^{dK} is an index for the concavity of the production function that is functionally the same as the coefficient of relative risk aversion. Full depreciation of the capital stock is a well-known special case that we will not pursue here (See, for instance, Romer 2006, p. 187). More generally, σ^{dK} is a parameter that indicates substitution possibilities between capital and labor. It depends on the production function and on the value of K in the steady state.

If our conjecture is correct, we can retrieve an expression for θ_K, the coefficient of proportionality between $\hat{c}_{y,s}$ and $\hat{K}_{s+1}$. Any part of the shock in period t that goes into K_{t+1} leads to a change in the resource constraint for period $t + 1$; analogous to (6), it reads

$$a_{cy}\hat{c}_{y,t+1} + a_{co}\hat{c}_{o,t+1} + a_{KS}\hat{K}_{t+2} = (1 + \bar{r})a_{KS}\hat{K}_{t+1}. \tag{13}$$

We can again simplify this using (7) and the conjecture:

$$\left(a_{co} + \frac{\sigma^o}{\sigma^y}(a_{cy} + a_{KS}\theta_K)\right)\hat{c}_{o,t+1} = (1 + \bar{r})a_{KS}\theta_K\hat{c}_{y,t}.$$

Now use this expression for $\hat{c}_{o,t+1}$ and the conjectured value for $\hat{K}_{t+1}$ in (11) to find

$$\sigma^y \hat{c}_{y,t} = \sigma^{dK}\theta_K\hat{c}_{y,t} + \frac{\sigma^o a_{KS}(1 + \bar{r})\theta_K}{a_{co} + \frac{\sigma^o}{\sigma^y}(a_{cy} + a_{KS}\theta_K)}\hat{c}_{y,t}$$

which has to be true for all $\hat{c}_{y,t}$. Some tedious manipulations show that θ_K is the solution to a quadratic equation,

$$\theta_K^2\left[a_{KS}\frac{\sigma^{dK}\sigma^o}{\sigma^y}\right] + \theta_K\left[\sigma^{dK}\left(a_{cy}\frac{\sigma^o}{\sigma^y} + a_{co}\right) + a_{KS}\sigma^o\bar{r}\right] - [\sigma^o a_{cy} + \sigma^y a_{co}] = 0. \tag{14}$$

Since $\sigma^{dK} > 0$ for any convex production technology and $\sigma^y, \sigma^o > 0$ due to diminishing marginal utility, this equation always has two (real) roots. It is possible to simplify this equation in the case of CRRA utility, when $\sigma^o = \sigma^y$. In that case,

$$\theta_K^2(1 - s^c)\sigma + \theta_K(\sigma s^c + (1 - s^c)\bar{r}) - s^c = 0$$

Table 1: Values for the coefficients of the policy function.

$\kappa \quad = a_{cy} + \frac{\sigma^o}{\sigma^y} a_{co} + \theta_K a_{KS}$

	$\pi_{.,A}$	$\pi_{.,\zeta}$	$\pi_{.,K}$
$\pi_{cy,.}$	a_Y/κ	$-a_\zeta/\kappa$	$(1+\bar{r})a_{KS}/\kappa$
$\pi_{co,.}$	$\frac{\sigma^y}{\sigma^o}\pi_{cy,A}$	$\frac{\sigma^y}{\sigma^o}\pi_{cy,\zeta}$	$\frac{\sigma^y}{\sigma^o}\pi_{cy,K}$
$\pi_{K_{+1},.}$	$\theta_K \pi_{cy,A}$	$\theta_K \pi_{cy,\zeta}$	$\theta_K \pi_{cy,K}$

with s^c the consumption share $a_{cy} + a_{co}$ and $\sigma = \sigma^{dK}/\sigma^y$. This shows that θ_K depends on the curvature of the production function relative to that of utility, on the investment share and on capital productivity. We can also deduce that there will be one positive root and one negative root. From an economic perspective, it is clear that we need the positive value for θ_K; below it turns out that the negative value leads to an unstable solution.

We summarize the state of affairs. The steady state of the planner's optimum can be computed as the solution set [$\bar{c}_y$,$\bar{c}_o$, $\bar{K}$, θ_K, $\bar{r}$] to the system of five equations (1), (3), (4), (10), and (14). The dynamics of the endogenous variables are summarized by the policy function

$$\hat{x} = \pi_{x,A}\omega_A + \pi_{x,\zeta}\omega_\zeta + \pi_{x,K_{-1}}\hat{K}$$

with $x \in \{c_y, c_o, K_{+1}\}$. These π's are related through the results above; their values are in Table 1.

Optimal dynamics in response to shocks in productivity and the rate of depreciation have two simple characteristics. Firstly, contemporaneous old and young generations have a proportional response. Bohn (2009) stresses that within the large class of CRRA utility functions, old and young generations should adapt their consumption equally much as $\sigma^y = \sigma^o$. Secondly, a fixed portion of the shock is moved to the future. The parameter that measures this is

$$\lambda \equiv \frac{\hat{c}_{y,s+1}}{\hat{c}_{y,s}} = \theta_K \pi_{cy,K}.$$

Below, I will refer to λ as the propagation parameter. If it is zero, none of the effects of any shock are felt outside the period in which it occurs. Higher values indicate that there is more intergenerational sharing. Diminishing marginal utility and concavity of production imply that $\lambda < 1$ so that the effect eventually dies out, as discussed in section 3.1.

3.4 Numerical implementation

Operationalize the utility- and the production functions as CRRA and Cobb-Douglas, respectively, so

$$u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}$$

and

$$F(K_t, n_t) = K_t^{1-\alpha} n_t^{\alpha}.$$

I write the conditions for the steady state so that each equation has one endogenous variable on the left-hand side.

$$\begin{aligned}
\bar{c}_y &= \frac{1}{n}\left(\bar{A}\bar{K}^{1-\alpha}n^{\alpha} - \bar{\zeta}\bar{K}\right) - \bar{c}_o \\
\bar{c}_o &= \mu^{\frac{-1}{\gamma}}\bar{c}_y \\
\bar{r} &= \frac{1}{\mu\beta\exp(\mathrm{Var}(\phi)/2)} - 1 \\
\bar{K} &= n\left[\frac{\bar{r}+\bar{\zeta}}{(1-\alpha)\bar{A}}\right]^{\frac{-1}{\alpha}} \\
\theta_K &= \frac{-b+\sqrt{b^2-4ac}}{2a}
\end{aligned}$$

where the variables a, b, and c are the coefficients of θ_K^2, θ_K and 1 in (14), respectively.

For a numerical example, take $\gamma = 2$, $\alpha = 0.75$, $\beta = 0.8$, $\mu = 1$, $\bar{A} = 1$ and $\sigma_A^2 = 0.04$, $\bar{\zeta} = 0.4$ and $\sigma_\zeta^2 = 0.01$ and $n = 1$. This leads to $\theta_K \approx 2.05$ or $\theta_K \approx -4.80$. The negative value for θ_K gives rise to a feasible steady state but makes the model unstable, leading to ever larger values for $\hat{K}$ when a shock materializes.[3] The positive θ_K leads to $\lambda \approx 0.604$.

The degree to which shocks are shared with future generations obviously depends on the discount rates β and μ; there is also a tradeoff between the desire to smooth consumption and the costs of straying too far from the optimal capital-labor ratio. The strengths of these two motives depend on utility parameter γ (a higher value leads to more intertemporal substitution) and labor share α (lower values make it easier to substitute capital for labor and facilitate intertemporal substitution). Table 2 shows how different values for α and λ influence intergenerational risk sharing.

[3] It turns out that the model is stable if θ_K obeys

$$\frac{a_{cy} + \frac{\sigma^y}{\sigma^o}a_{co}}{-(2+\bar{r}\,)a_{KS}} < \theta_K < \frac{a_{cy} + \frac{\sigma^y}{\sigma^o}a_{co}}{\bar{r}a_{KS}}$$

noting that the a's and $\bar{r}$ also depend on θ_K.

Table 2: Values for the propagation parameter λ.

	$\alpha = 0.7$	$\alpha = 0.75$	$\alpha = 0.8$
$\gamma = 0.8$	0.487	0.433	0.372
$\gamma = 1$	0.529	0.475	0.414
$\gamma = 2$	0.653	0.604	0.547
$\gamma = 3$	0.717	0.674	0.622
$\gamma = 6$	0.809	0.776	0.735

For a slightly different perpective on the same issue. we compute the fraction of the shock that each generation receives. The change in gross output (production plus scrap capital) due to the shock is absorbed by different generations who change their consumption. If we discount consumption changes by $\bar{r}$, the rate of interest at the steady state, the present value of the consumption changes must be (approximately) equal to the change in output. Figures 1(a) and 1(c) on page 19 show the distribution of the two shocks under optimal control. The year of birth of each generation is on the horizontal axis, with the shock arriving at time t. We will compare these distributions to those obtained in a market setting.

3.5 Perturbation methods

To assess the validity of the conjecture that $\hat{K}_{s+1}$ is a linear function of $\hat{c}_{y,s}$, I solve the model in Dynare++ and use the impulse-response coefficients of K and c_y to find the optimal ratio of the log-deviations $\hat{K}_{s+1}$ and $\hat{c}_{y,s}$. I report the results for $s = t, \ldots, t + 10$, noting that as reponses die out the numerical approximations of this ratio become more unreliable. The results are in table 3; for the two types of shocks (each of one standard deviation), they differ somewhat, especially in the early periods. When assessing whether the series come close enough to the conjecture that θ_K is a constant, keep in mind that they too are the result of a numerical procedure that approximates the true optimal response to shocks.

In both cases, the optimal response of capital is a bit higher than the log-linearized model suggested, and there is some variation over time in the ratio of $\hat{K}_{s+1}$ to $\hat{c}_{y,s}$. These fluctuations do stay within 10% of the derived value of $\theta_K = 2.05$.

Table 3: Numerical results for $\theta_K = \hat{K}_{s+1}/\hat{c}_{y,s}$ from the solution obtained through perturbation methods.

	t	$t+1$	$t+2$	$t+3$	$t+4$	$t+5$	$t+6$	$t+7$
ω_A	2.097	2.142	2.203	2.255	2.252	2.158	2.071	2.043
ω_ζ	2.034	2.084	2.173	2.230	2.245	2.166	2.068	2.046

	$t+8$	$t+9$	$t+10$
ω_A	1.964	2.025	2.036
ω_ζ	1.961	2.021	2.032

4 Market equilibrium

We maintain the assumptions of the previous section but compute the decentralized solution, in which agents maximize their own utility and firms maximize profits. The latter are characterized by competitive markets due to the production function, which exhibits constant returns to scale. Hence wages and the return to capital are set, as usual, at the margin.

4.1 Setup

4.1.1 Agents

As before, agents live in two periods. In their *young* period, they supply labor inelastically and receive a wage income. They can save by buying capital. In their *old* period, agents become passive consumers, using whatever income they arranged in the previous period. Those who invested in one unit of capital in the previous period receive a compensation that is equal to

$$A_t \cdot F_K(K_t, n_t) + 1 - \zeta_t$$

or the marginal product of capital plus whatever is left after depreciation. The rest of production goes to labor, so that the wage is

$$w_t = A_t \cdot F_n(K_t, n_t).$$

Note that the return to capital is uncertain at the time the investment takes place, due to the stochastic nature of A_t and ζ_t.

The young generation in each period has to decide how much resources to move to the next period, in excess of existing pension schemes. It can do this by investing in capital.

This environment leads to two budget constraints, which each agent must obey. When young, there must hold

$$c_{y,t-1} = w_{t-1} - k_t \tag{15}$$

which says that consumption must equal net wage income minus capital purchased.Old agents are subject to

$$c_{o,t} = (1 + r_t)k_t. \tag{16}$$

The stochastic elements in this equation are in r_t, which contains $\omega_{A,t}$ and $\omega_{\zeta,t}$. Both are revealed at time t.

4.2 Equilibrium

Agents maximize utility by conforming to the Euler equation

$$u'(c_{y,s-1}) = \beta \mathrm{E}_{s-1}((1 + r_s^r)u'(c_{o,s})) \tag{17}$$

for each s.

4.2.1 Inactive government

Consider first the situation in which the government is absent from the model and there is no pension system. Agents only have to decide their holdings of capital. Again, we look first at the median system. Writing (17) as

$$u'(c_{y,s-1}) = \beta(1 + \bar{r})u'(\bar{c}_o)\mathrm{E}_{s-1}(\exp(\hat{r}_s - \sigma^o \hat{c}_{o,s})) \tag{18}$$

as before, note that the stochastic process for $\hat{r}_s$ is unchanged from (8). But the changes in marginal utility of the old generation are different in this case. Since there is no redistribution, part of the shock in productivity and the complete shock in depreciation is absorbed fully by the holders of capital, the old generation. This means that

$$\hat{c}_{o,s} = a_{Ko}\omega_{A,s} - a_{\zeta o}\omega_{\zeta,s} \tag{19}$$

with $a_{Ko} = \bar{A}F_K(\bar{K}, n)\bar{K}/(n\bar{c}_o)$ and $a_{\zeta o} = \bar{\zeta}\bar{K}/(n\bar{c}_o)$. Write $\phi_{K,s} = \hat{r}_s - \sigma^o \hat{c}_{o,s}$ to find

$$\phi_{K,s} = (a_K - \sigma^o a_{Ko})\omega_{A,s} - (a_{\zeta K} - \sigma^o a_{\zeta o})\omega_{\zeta,s}$$

so that the expectation in (18) becomes

$$\begin{aligned}\mathrm{E}_{s-1}(\exp(\hat{r}_s + \hat{y}_s)) &= \exp\left[\frac{1}{2}\mathrm{var}(\phi_{K,s})\right] \\ &= \exp\left[\frac{1}{2}\left((a_K - \sigma^o a_{Ko})^2\sigma_A^2 + (a_{\zeta K} - \sigma^o a_{\zeta o})^2\sigma_\zeta^2\right)\right]\end{aligned}$$

This defines the steady state of the model.

As for the dynamics, shocks in ζ, the rate of depreciation, are fully absorbed by the current old generation. This implies that there is no impact after the period in which the shock takes place. Shocks in productivity A are felt by the old generation through their capital income and by the young through their wage income. The latter can choose to modify their current consumption as well as their investment in capital. Through capital, the shock has repercussions in the periods after it takes place.

Consider the saving decision of the young. A change in savings will lead to different income in the old stage, due to both higher capital holdings and a lower rate of return.[4] We write

$$\Delta c_{o,t+1} = (1 + \bar{r}\ )\Delta K_{t+1} + \Delta r_{t+1}\bar{K}, \tag{20}$$

ignoring the cross-product of the changes which is likely to be small. Simplify this as

$$\begin{aligned}\hat{c}_{o,t+1} &= \left((1 + \bar{r}\ )\hat{K}_{t+1}\bar{K} + (1 + \bar{r}\ )\hat{r}_{t+1}\bar{K}\right)/\bar{c}_o \\ &= (1 + \bar{r}\ )\frac{\bar{K}}{\bar{c}_o}(\hat{K}_{t+1} - \sigma^{dK}\hat{K}_{t+1}) \\ &= (1 - \sigma^{dK})\hat{K}_{t+1}\end{aligned} \tag{21}$$

where the final step follows from the income identity $\bar{c}_o = (1 + \bar{r})\bar{K}$.

Conjecture again that $\hat{K}_{t+1} = \theta_K\hat{c}_{y,t}$. Using the above and the conjecture in the log-linearized Euler condition (11) gives

$$\sigma^y\hat{c}_{y,t} = \sigma^{dK}\theta_K\hat{c}_{y,t} + \sigma^o(1 - \sigma^{dK})\theta_K\hat{c}_{y,t}$$

which has to hold for all $\hat{c}_{y,t}$. This gives θ_K as

$$\theta_K = \frac{\sigma^y}{\sigma^{dK} + (1 - \sigma^{dK})\sigma^o}. \tag{22}$$

[4] Notice the difference between this expression and the analogous formula in the command optimum, (13). In the current market setting, the lowering of the rate of return entails a transfer from the old to the young generation, as it applies to all of their capital holdings. I assume that the old generation takes this transfer into account when optimizing. In the command optimum, the division of the production is not affected by the lower marginal product of capital as it is decided by the planner.

Table 4: Values for the coefficients of the policy function under *laissez faire*.

$\kappa_m = a_{cy} + \theta_K a_{KS}$

	$\pi_{.,A}$	$\pi_{.,\zeta}$	$\pi_{.,K}$
$\pi_{cy,.}$	a_L/κ_m	0	$\sigma^{dK}(1+\bar{r})a_{KS}/\kappa_m$
$\pi_{co,.}$	a_{Ko}	$-a_{\zeta o}$	$1-\sigma^{dK}$
$\pi_{K_{+1},.}$	$\theta_K \pi_{cy,A}$	0	$\theta_K \pi_{cy,K}$

The budget constraint for the young is a variation on (6)

$$a_{cy}\hat{c}_{y,t} + a_{KS}\hat{K}_{t+1} = a_L \omega_{A,t} + \sigma^{dK}(1+\bar{r})a_{KS}\hat{K}_t \tag{23}$$

with $a_L = \bar{A}F_n(\bar{K}, n)n/\bar{T}$, the labor share. The impact of changes in the capital stock follows from the way the wage sum $w_t n_t = A_t[F(K_t, n_t) - F_K(K_t, n-t)]$ changes when capital changes: $\mathrm{d}(wn) = -\bar{A}F_{KK}\bar{K}\mathrm{d}K$. This (and the conjecture) gives the impact of shocks in A and K on c_y, from which the other dynamics follow. We can summarize these results in a policy function; the coefficients are in Table 4, which can be compared to the optimal coefficients in Table 1 above.

The two tables show that the way shocks are absorbed in the market economy is different from the social optimum. The old and young generations generally do not share optimally in the shocks' effects. The extent of intergenerational risksharing depends on the curvature of the production function, a factor that did not enter into consideration in the optimal solution.

Numerical implementation of the median system

We solve this model using the same parameters as in section 3.4. The system now has four variables ($\bar{c}_y$, $\bar{c}_o$, $\bar{K}$ and $\bar{r}$) and four equations (rewritten from 3, 4, 16 and 18):

$$\begin{aligned}
\bar{c}_y &= \frac{1}{n}\left(\bar{A}\bar{K}^{1-\alpha}n^{\alpha} - \bar{\zeta}\bar{K}\right) - \bar{c}_o \\
\bar{K} &= \left(\frac{\bar{r}+\bar{\zeta}}{\bar{A}(1-\alpha)n^{\alpha}}\right)^{\frac{-1}{\alpha}} \\
\bar{c}_o &= (1+\bar{r})\bar{K} \\
\bar{r} &= \frac{(\bar{c}_y/\bar{c}_o)^{-\gamma}}{\beta\exp\left[\frac{1}{2}\left((a_K - \sigma^o A_{Ko})^2\sigma_A^2 + (a_{\zeta K} - \sigma^o A_{\zeta o})^2\sigma_\zeta^2\right)\right]} - 1
\end{aligned} \tag{24}$$

Table 5: Numerical solution of the steady state

	Command opt.	Market	change
Capital	0.288	0.224	−22.2%
Production	0.732	0.688	−6.1%
Depreciation	0.115	0.090	−22.2%
Investment / production	0.157	0.130	−17.1%
Consumption young	0.309	0.292	−5.4%
Consumption old	0.309	0.306	-0.7%

Table 6: Values for the propagation parameter λ in the market economy.

	$\alpha = 0.7$	$\alpha = 0.75$	$\alpha = 0.8$
$\gamma = 0.8$	0.284	0.236	0.190
$\gamma = 1$	0.300	0.250	0.200
$\gamma = 2$	0.345	0.284	0.223
$\gamma = 3$	0.366	0.298	0.232
$\gamma = 6$	0.389	0.312	0.240

The production technology and the distribution of the shocks are the same as in section 3.4, but the outcome is different. Because the shocks are now shouldered mainly by the old, the proposition of investing in capital is less attractive. We find that, using the same parameters, the capital stock is 24% lower and production 6.5% lower. In the steady state, old people expect to consume 5% more than young people, whereas previously consumption was constant. See Table 5 for details.

As for dynamics, $\theta_K \approx 1.27$, compared to our previous 2.05 for the standard parameters. This indicates that the young generation moves a smaller portion of the shock to future periods. This also follows from the propation parameter λ, values for which are in Table 6. Compared to Table 2, for the same parameters, they are much closer to zero.

We again compute the fraction of the shock that each generation receives by discounting consumption changes by $\bar{r}$, the rate of interest at the steady state. The present value of all consumption changes must be (approximately) equal to the change in output. Figure 1 shows the distribution of the two shocks under optimal control and the decentralized equilibrium. The year of birth of each generation is on the horizontal axis, with the shock arriving at time t.

Both the tables and the figure show that a decentralized equilibrium concentrates the effects of a shock among the generations alive at the time it hits. The rest of this paper analyzes different pension institutions that will affect the distribution of the shocks. Our main question is whether it is possible to emulate the distribution that arises under the command optimum, a question that is addressed in section 5.

4.2.2 Pay-as-you-go

Agents are enrolled by default in a pay-as-you-go (PAYGO) scheme in which they must pay an amount $\theta^p + \theta^w w_{t-1}$ when young, indicating that there may be both lump-sum and wage-related contributions. The payout to old agents is $(n_t/n_{t-1})(\theta^p + \theta^w w_t)$. The parameters θ^p and θ^w are set by the government and we assume that they are constant.

Without demographic uncertainty, a nonzero θ^p has the effect of changing the budget constraints for the two generations without changing the (absolute) level of risk exposure. Setting θ^w to values other than zero transfers some of the period's wage uncertainty to the old generation.

As above, the main effect can most easily be seen on $\hat{c}_{o,s}$. Shocks to productivity and depreciation have an impact of

$$\hat{c}_{o,s} = a_{PG}\omega_{A,s} - A_{\zeta o}\omega_{\zeta,s} \tag{25}$$

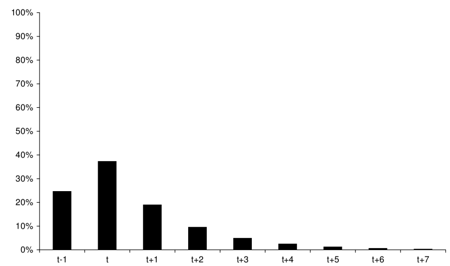
(compare formula 19) with $a_{PG} = \bar{A}((1-\theta^w)F_K(\bar{K},n)\bar{K} + \theta^w F(\bar{K},n))/(n\bar{c}_o)$. This constant is larger than a_{Ko}, which it replaces in the relevant equations.

Apart from unforeseen shocks, changes in the capital stock also impact on old-age consumption. We pick up from formula (20) above, which changes to

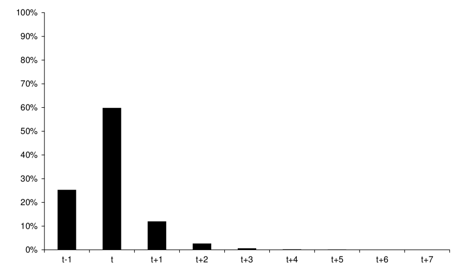
$$\Delta c_{o,t+1} = (1+\bar{r}\ )\Delta K_{t+1} + \Delta r_{t+1}\bar{K} + \theta^w \bar{n}\Delta w_{t+1}.$$

Writing total wages as $A[F(K) - KF_K(K)]/n$, their derivative with respect to capital can be expressed as $(1+\bar{r}\ )\sigma^{dK}/n$. Using this to approximate Δw_{t+1} above, we write

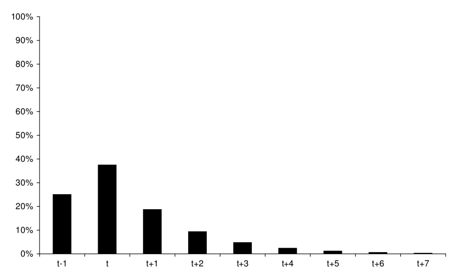
$$\begin{aligned}
\Delta c_{o,t+1} &= (1+\bar{r}\ )\left(1-(1-\theta^w)\sigma^{dK}\right)\Delta K_{t+1} \\
\hat{c}_{o,t+1} &= \frac{(1+\bar{r}\ )\bar{K}}{\bar{c}_o}\left(1-(1-\theta^w)\sigma^{dK}\right)\hat{K}_{t+1} \\
&\equiv a_{ro}\left(1-(1-\theta^w)\sigma^{dK}\right)\hat{K}_{t+1}
\end{aligned} \tag{26}$$



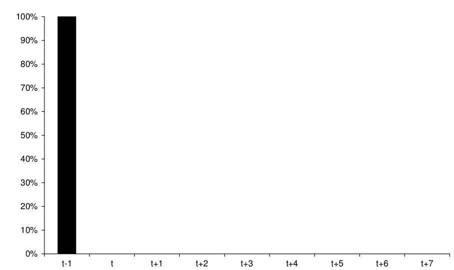
(a) Generational distribution of $\omega_{A,t} = 0.2$, command optimum.



(b) Generational distribution of $\omega_{A,t} = 0.2$, decentralized equilibrium.



(c) Generational distribution of $\omega_{\zeta,t} = 0.1$, command optimum.



(d) Generational distribution of $\omega_{\zeta,t} = 0.1$, decentralized equilibrium.

Figure 1: The distribution of different shocks over the generations (indexed by year of birth) in two arrangements.

Table 7: Values for the coefficients of the policy function with a PAYGO system.

$\kappa_m = a_{cy} + \theta_K a_{KS}$

	$\pi_{\cdot,A}$	$\pi_{\cdot,\zeta}$	$\pi_{\cdot,K}$
$\pi_{cy,\cdot}$	$(1-\theta^w)a_L/\kappa_m$	0	$(1-\theta^w)\sigma^{dK}(1+\bar{r})a_{KS}/\kappa_m$
$\pi_{co,\cdot}$	a_{PG}	$-a_{\zeta o}$	$a_{ro}(1-(1-\theta^w)\sigma^{dK})$
$\pi_{K_{+1},\cdot}$	$\theta_K \pi_{cy,A}$	0	$\theta_K \pi_{cy,K}$

with a_{ro} the share of capital income for the old generation.[5] From this and the Euler condition (11), we compute θ_K, which is now

$$\theta_K = \frac{\sigma^y}{\sigma^{dK} + \sigma^o a_{ro}(1-(1-\theta^w)\sigma^{dK})}. \tag{27}$$

Here we can see the effect of introducing a PAYGO system on the accumulation of capital by individuals, as it works through the riskiness of their income. This effect is separate from the well-known replacement effect mentioned above.[6] Setting θ^p to a nonzero value decreases a_{ro} here, and thus increases θ_K. By providing a safe income in the second period, a lump-sum PAYGO system increases the demand for (risky) capital in the first period. Setting θ^w to a nonzero value instead does two things to θ_K: not only does a_{ro} decrease, but θ^w goes up. The net effect is ambiguous. This is because a wage-linked PAYGO-system introduces income, but also extra risk in the second period.

As before, we write the budget constraint for the young (compare formula 23)

$$a_{cy}\hat{c}_{y,t} + a_{KS}\hat{K}_{t+1} = (1-\theta^w)a_L\omega_{A,t} + (1-\theta^w)\sigma^{dK}(1+\bar{r})a_{KS}\hat{K}_t \tag{28}$$

and use it to construct Table 7.

[5]Without PAYGO, this share is one and $\theta^w = 0$, which brings us back to (21). Conversely, if the PAYGO system gives the old generation all of production, $\theta^w = 1$ and the term σ^{dK} disappears as the transfer from capital- to labor-income becomes irrelevant. See also footnote 4.

[6]Krueger and Kubler (2006) separate the effect of decreased riskiness on welfare into two (p. 747): the *consumption insurance effect* is the increased utility from the lower variability of old-age income; the *effect on mean consumption* is the added gain (or loss) from changing the portfolio in response to this new asset.

Table 8: Numerical solution of the steady state

	Com. opt.	Market	Paygo $\theta_p = 0.05$		Paygo $\theta_w = 0.106$	
Capital	0.288	0.224	0.156	(−30.2 %)	0.157	(−29.8%)
Production	0.732	0.688	0.629	(−8.6%)	0.630	(−8.5%)
Depreciation	0.115	0.090	0.063	(−30.2%)	0.063	(−29.8%)
Inv. / prod.	0.157	0.130	0.100	(−23.6%)	0.100	(−23.3%)
Cons. young	0.309	0.292	0.265	(−9.2%)	0.265	(−9.2%)
Cons. old	0.309	0.306	0.301	(−1.7%)	0.302	(−1.5%)

Numerical implementation of the median system

We solve this model using the same parameters as in section 3.4. The first two equations are the same as those in (24), the other two are

$$\begin{aligned} \bar{c}_o &= (1+\bar{r})\bar{K} + \theta^p + \theta^w \bar{A}\alpha n^{\alpha-1}\bar{K}^{1-\alpha} \\ \bar{r} &= \frac{(\bar{c}_y/\bar{c}_o)^{-\gamma}}{\beta \exp\left[\frac{1}{2}(a_K - \sigma^o A_{PG})^2\sigma_A^2 + \frac{1}{2}(a_{\zeta K} - \sigma^o A_{\zeta o})^2\sigma_\zeta^2\right]} - 1. \end{aligned} \tag{29}$$

Introducing a PAYGO system has the well-known effect of decreasing the steady-state capital stock (the replacement effect). The new steady-state outcome is in table 8, where the percentages indicate the difference with the market (and inactive government) equilibrium. The two columns show the effects of a small PAYGO system that works either lump-sum or through a tax on labor. The latter tax is set so that the amount transferred is the same as the lump sum, in the median system. Note that the differences between the two forms of PAYGO are very slight.

The distribution of shocks is virtually the same as above. For shocks in the rate of depreciation, the owners of capital still pick up the check. With a lump sum tax, productivity shocks are also not affected. With a tax on labor, a larger share of productivity shocks now falls on the old generation. This does not insulate the young generation, however, since the tax is proportional and simply takes away part of their income.

If it is the riskiness of capital that leads young people to accumulate too little of it (compared to the command optimum), the current instrument may fix this problem if it is applied in a non-orthodox way: setting $\theta^w = -0.1$ approximately returns the economy to the efficient level of capital. Being faced with an endowment when young, but an unavoidable tax when old, unsurprisingly leads people to save more. This receive-as-you-go system does nothing to fix the inefficient division of the shocks, however.

As for the dynamics, $\theta_K = 1.48$ when $\theta^p = 0.05$, up from 1.27 in the laissez-faire case. In this case, $\lambda = 0.351$. Replacing this system with a similar-sized, but wage-linked PAYGO with $\theta^w = 0.106$ gives $\theta_K = 1.39$ and $\lambda = 0.304$. These numbers illustrate the exposition in the previous paragraph.

4.2.3 Funded pensions

To the PAYGO system we add a mandatory funded pension system, in which premiums are collected lump-sum. This means that the system is defined by the rules determining payouts. I ignore the DC-system which is shown to add nothing to the existing menu in Beetsma and Bovenberg (2006) and concentrate on what they refer to as defined real benefits (DRB). In this case, the benefits are a certainty in terms of consumption goods.

The pension fund uses the (time-invariant) premiums to buy assets. In a per-capita notation,

$$\theta^f = k^f_{t-1}. \tag{30}$$

The fund is actuarially fair, so that the promised rate of return to members is $r^f_t = \mathrm{E}_{t-1}(r_t)$. However, the actual rate of return that the fund gets on its capital is subject to shocks. The residual claim on the investment, $k^f_{t-1}(r_t - r^f_t)$, is absorbed by the young generation. Given that only one asset is available, the fund cannot completely avoid this mismatch risk. In a sense, the establishment of a pension fund allows next period's young to receive part of the random scrap value of capital, which is realized in the period of their youth. Furthermore, a DRB pension fund will reduce the variability of old-age income insofar as it substitutes for the holding of risky assets.

Call the private holdings of capital by the old generation K^p_t, so that $K_t = K^p_t + K^f_t$. The new expression for $\hat{c}_o$ (compare 25) is

$$\hat{c}_{o,s} = a_{PGP} \cdot \omega_{A,s} - a_{\zeta oP} \cdot \omega_{\zeta,s} \tag{31}$$

with $a_{PGP} = \bar{A}((1-\theta^w)F_K(\bar{K},n)\bar{K}^p + \theta^w(F(\bar{K},n) - F_K(\bar{K},n)\theta^f))/n\bar{c}_o$ and $A_{\zeta oP} = \zeta\bar{K}_p/n\bar{c}_o$. Increasing the size of the pension fund leads to smaller values for A_{PGP} and $A_{\zeta oP}$.

The effect of introducing pension funds, to a first approximation, is that the private holdings of capital will adjust in the opposite direction. The net effect of this is that a part of the capital holdings of the old generation has become riskless. This reduction of risk leads to a second order effect, namely a decrease in the desire to save as precautionary savings fall.

To find the steady state of the model we again use the Euler equation (11) so we need the effect of private capital investments on old-age consumption. Correcting for the fact that there are now two holders of capital, this is the same as (26):

$$\hat{c}_{o,t+1} = a_{ro}\left(1-(1-\theta^w)\sigma^{dK}\right) s_k^p \hat{K}_{t+1}^p$$

with $s_k^p = K_t^p/K_t$. It may be somewhat surpising that the introduction of a funded pension system does not materially change this expression. The reason for this is that the rate of return that is promised on the funded pensions, r_t^f, is equal to the expected rate of return at the time of the contribution. If the young generation decides to increase investment this drives down the expected return on both private investments and the funded system. This affects their decision just as if the pension fund's capital were held by the young themselves.

The value of θ_K is the same as before, see (27). But note again the definition of this parameter: it is the coefficient of proportionality between the log-change in c_y and the *total* capital stock K. To get the coefficient between c_y and private capital k^p, we need to divide θ_K by private capital's share s_k^p.

All this makes that the median state of this model is very similar to that of the laissez-faire situation of section 4.2.1 (provided that we set the PAYGO parameters to zero for the moment). The only change is in the variability of old-age income, which has an effect on the saving decision because it changes the correlation between $\hat{c}_o$ and $\hat{r}$. This effect is, as we shall see, small. The reason that these two models are so much alike is that though the private holdings of physical capital differ, the marginal decision to acquire it is almost the same.

The budget constraint for the young gets an extra term compared to formula (28).

$$\begin{aligned} &a_{cy}\hat{c}_{y,t} + a_{KS}\hat{K}_{t+1} = \\ &\qquad ((1-\theta^w)a_L + a_{AyP})\omega_{A,t} + a_{\zeta yP}\omega_{\zeta,t} + (1-\theta^w)\sigma^{dK}(1+\bar{r})a_{KS}\hat{K}_t \end{aligned} \tag{32}$$

(with $a_{AyP} = \bar{A}F_K(\bar{K},n)\theta^f/\bar{T}$ and $a_{\zeta yP} = \bar{\zeta}\theta^f/\bar{T}$) after which we use it to construct Table 9.

Note that the funded pension system mainly works to shift risk from the old to the young generation. For the depreciation risk, which was concentrated with the old generation under the market system, this leads to more risk-sharing. But for the productivity risk, it is not clear that the young generation, which is already exposed through wage fluctuations, needs this extra risk.

Table 9: Values for the coefficients of the policy function with a PAYGO system and a funded pension system.

$$\kappa_m = a_{cy} + \theta_K a_{KS}$$

	$\pi_{.,A}$	$\pi_{.,\zeta}$	$\pi_{.,K}$
$\pi_{cy,.}$	$((1-\theta^w)a_L + a_{AyP})/\kappa_m$	$-a_{\zeta yP}/\kappa_m$	$(1-\theta^w)\sigma^{dK}(1+\bar{r})a_{KS}/\kappa_m$
$\pi_{co,.}$	a_{PGP}	$-a_{\zeta oP}$	$a_{ro}(1-(1-\theta^w)\sigma^{dK})$
$\pi_{K_{+1},.}$	$\theta_K \pi_{cy,A}$	$\theta_K \pi_{cy,\zeta}$	$\theta_K \pi_{cy,K}$

Table 10: Numerical solution of the steady state

	Com. opt.	Market	Funded $\theta^f = 0.1$		Funded $\theta^f = 0.2$	
Capital	0.288	0.224	0.223	(−0.3%)	0.224	(−0.1%)
Production	0.732	0.688	0.687	(−0.1%)	0.688	(0.0%)
Depreciation	0.115	0.090	0.089	(−0.3%)	0.090	(−0.1%)
Inv. / prod.	0.157	0.130	0.130	(−0.2%)	0.130	(−0.1%)
Cons. young	0.309	0.292	0.292	(+ 0.1%)	0.292	(0.0%)
Cons. old	0.309	0.306	0.306	(−0.2%)	0.306	(−0.1%)
Interest rate	0.236	0.368	0.369		0.368	

Numerical implementation of the median system

The system in this case is the same as in (24), with the one change that A_{PG} is replaced by A_{PGP} and $A_{\zeta o}$ by $A_{\zeta oP}$. We set paygo-parameters to zero for now. The results are in table 10.

Note that differences with the market solution are slight, even though in the last columns the funded pension system owns close to 90% of the capital stock. As predicted above, the effects are largest when the funded system owns about half the capital stock, which is when the variability in old-age income and in r just cancel each other out.

5 An optimal pension system

Beetsma and Bovenberg (2006) show that it is possible to configure a pension system with paygo and DRB such that the distribution of the shocks over different generations is the same as in the command optimum. The question remains whether this result can be replicated in a setting with many periods. In this section, we gather the results of the previous analysis to provide an answer to this question.

5.1 Degrees of freedom

To get the same distribution of shocks in the command optimum and in a market system with PAYGO and funded pensions, we need to make sure that the coefficients of the policy function (the π's) are the same in both cases. This means that the entries of Tables 1 and 9 have to be exactly equal. At first glance, this gives nine equations and so nine instruments would be needed to make this happen. Fortunately, we can reduce this number somewhat. Firstly, the resource constraint guarantees that in each column, if the first two elements match, then the third will automatically match as well. This leaves six equations.

Secondly, note that in both cases, the values of $\pi_{cy,x}$ and $\pi_{K_{+1},x}$ have the same ratio for each $x \in \{A, \zeta, K\}$. (Admittedly, this is true by our conjecture. But this conjecture was tested in section 3.5 above). This means that all these three equations are true if one of them is true. This reduces the number of equations to four.

Looking at the π's themself, it is intuitive to see what these four equations represent. The log-change of the consumption of the young and the old generation has to differ by a fixed factor of σ^y/σ^o in the optimal case. With a CRRA utility function, this gives the stronger result that these responses have to be equal. In the decentralized equilibrium, there is no fixed proportion between $\hat{c}_y$ and $\hat{c}_o$; rather, ownership of the different factors (possibly redistributed by the government or the funded pension scheme) determines the response. Getting these consumption responses in line requires that for each of the three shocks, an equation has to be satisfied. The fourth condition is that θ_K is the same in both cases.

The main result of Beetsma and Bovenberg (2006) is that a pension system with the characteristics that we have used so far can replicate the command optimum if the parameters are set just right. They find this result in a somewhat special economy in which there are only three periods, the first and the last of which are endowment economies. Our efforts so far show that this result cannot be replicated in a more general environment with infinitely many periods and two overlapping generations. This is because there are only three free parameters (θ^f, θ^p and θ^w) and four conditions that have to be satisfied.

In general, there will be no solution that equates all parameters of the policy function.

Another way to phrase this problem is that we need an extra instrument to deal with the intertemporal allocation of shocks. In the command optimum, the central planner decides how to allocate resources and (until this allocation has taken place) is the sole owner of those resources. In a market equilibrium, ownership of production factors is well defined, and with it the ownership of factor payments. The government may redistribute between generations, as it does with the PAYGO system, but until now it has not directly moved resources between periods. The only way to achieve this, in the current model, is by investing in physical capital. This is something that has been left to the (young) agents and to the pension fund, with the latter not responding to shocks. In that case, the propagation of any shock through time will have to go through the investment decision of the young.

When a generation invests in extra capital, it is itself the main beneficiary of this investment. There is an external effect to this transaction, however, which is to raise the capital-labor ratio and thus to raise wages for the future young. This effect can be quantified. Suppose that the current young increase their investment by an amount ϵ. This increases their future income by $(1-\sigma^{dK})(1+r)\epsilon$. The (wage-) income of the next young generation is increased by $\sigma^{dK}(1+r)\epsilon$. The coefficient σ^{dK} is defined on page 9 and indicates the curvature of the production function at the current point, a measure of how much changes in the capital-labor ratio will affect wages and rents.

In the simulations run in this paper, $\sigma^{dK}(1+r)$ is around 0.55. This means that even if the young generation invests *all* of the proceeds from a positive shock in capital, slightly more than half of them will go to the next generation. In contrast, the optimal scenario in section 3 has a propagation parameter of 0.6, which requires that the young pass on about as much as their own change in consumption to generations unborn at the time of the shock.

5.2 Funded pensions as transmission mechanism

One way to transfer a larger part of any shock to the future is to use funded pensions as transmission mechanism. Currenty, these pensions take a fixed amount θ^f from the young generation and guarantee the rate of return that is expected by the market. Any shocks that materialize between the collection of the premiums and the payout of the benefits are absorbed by the young generation.

One way of introducing an extra instrument and an extra degree of freedom is to allow pension fund to retain part of the mismatch on their balance to distribute to following generations. This opens up a wider channel to get shocks transferred to the future. An alternative setup could only give the current young a fraction, say $0 < \theta^{\mathrm{pf}} \leq 1$, of the mismatch between assets and obligations, with the rest flowing into the fund's capital holdings. This opens up a gap between assets and obligations in the next period as well, at which time the *new* young generation could absorb another θ^{pf} of the gap. Asymptotically the gap is filled, provided that θ^{pf} is not too small.

The problem here is that the optimal policy function is very specific about the variables that can influence each other. The effect of capital on old-age consumption, $\pi_{co,K}$, is a fixed number (see table 9). In case of retention by the pension fund, there would be two different effects depending on whether the capital was saved by the agents in their young years or by the pension fund. This can only be expressed in a policy function by introducing an extra state variable; but we know that this cannot be optimal as the social planner uses only the present three state variables.

6 Conclusions

Pension institutions reallocate resources and risks between generations. This paper has analyzed these two functions by comparing their effects to the workings of an optimal central planner, an approach shared with Beetsma and Bovenberg (2006). In that work, the authors show that it is possible to emulate optimal risk-sharing by adjusting the parameters of the pension system. But where the analysis in the current paper looks at a dynamic economy with many generations, Beetsma and Bovenberg only consider a single period with two overlapping generations.

It turns out that allowing many generations alters the conclusions and brings different effects to the foreground. Firstly, the level of steady state production is affected by the type of pension arrangement. This is well known in the literature, where the effects of introducing a PAYGO system on the rate of investment has been studied extensively (see for instance Diamond 1965). The simulations in this paper show that the gaps between the different steady states are large, and should form a prominent part of the analysis. Both PAYGO pensions and the introduction of government bonds lead to a lower level of the steady state.

Secondly, optimal risk sharing is harder than may be thought on the basis of earlier analysis. The optimal system calls for broad insurance of the shocks over different generations but the only way to move resources over time (and to different generations) is by modifying the capital stock. In a *laissez-faire* equilibrium, the next generation feels the effect of the changed stock of capital in their wage rate. This channel, it turns out, may be too small to accomodate the intergenerational transfer that is necessary in the optimum. If a funded pension fund is allowed to respond to the shocks by letting its capital holdings vary, the transmission channel widens and it is possible to involve future generations in current shocks. It is in general not possible to distribute these shocks exactly as the central planner would, however. Also, for negative shocks future generations may not want to share, leading to discontinuity risk.

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A Derivations

A.1 Euler conditions

The Bellman equation associated with the planner's problem is

$$V(K_t, n_{t-1}, n_t, A_t, \zeta_t) = \max_{c_{o,t}, c_{y,t}} [n_{t-1}u(c_{o,t}) + n_t u(c_{y,t}) + \\ \beta \mathrm{E}_t V(A_t F(K_t, n_t) - c_{o,t} - c_{y,t} + (1-\zeta_t)K_t, n_t, n_{t+1}, A_{t+1}, \zeta_{t+1})]$$

where the expectation E_t is with respect to the two stochastic variables,

$$\mathrm{E}_t V(K_{t+1}, n_t, n_{t+1}, A_{t+1}, \zeta_{t+1}) = \int\int V(K_{t+1}, n_t, n_{t+1}, A_{t+1}, \zeta_{t+1})\,\mathrm{d}F(\zeta_{t+1})\,\mathrm{d}F(A_{t+1})$$

Here I use $\xi_t = \{A_t, \zeta_t\}$ as a shorthand, its (time-invariant) accumulated density function is $F(\xi_t)$.

Introduce the function

$$\begin{aligned}\mathbf{u}(K_t, K_{t+1}) =& \max_{c_{o,t}} n_{t-1}u(c_{o,t}) + n_t u(c_{y,t}) \\ c_{y,t} =& \frac{1}{n_t}(A_t F(K_t, n_t) + (1-\zeta_t)K_t - n_{t-1}c_{o,t} - K_{t+1})\end{aligned}$$

where the maximum is found using the intratemporal condition

$$u'(c_{o,s}) = u'(c_{y,s}) \qquad \forall s \geq t.$$

as in (1). We can then write the problem as

$$V(K_t, n_{t-1}, n_t, A_t, \zeta_t) = \max_{K_{t+1}}[\mathbf{u}(K_t, K_{t+1}) + \beta\mu \mathrm{E}_t V(K_{t+1}, n_t, n_{t+1}, A_{t+1}, \zeta_{t+1})] \tag{33}$$

Due to the structure of the problem, we can eliminate the value function here (see Ljungqvist and Sargent 2000, p. 37): take the derivative of (33) to K_t

$$\frac{\mathrm{d}V(K_t)}{\mathrm{d}K_t} = \left.\frac{\mathrm{d}\mathbf{u}(K_t, K_{t+1})}{\mathrm{d}K_t}\right|_{K_{t+1}=K^*_{t+1}}$$

where K^*_{t+1} is the optimal value in (33). This follows from the envelope theorem. Using the definition of $\mathbf{u}$, this is

$$\frac{\mathrm{d}V(K_t)}{\mathrm{d}K_t} = u'(c_{o,t})(A_t F_K(K_t, n_t) + 1 - \zeta_t) \tag{34}$$

Note also that the maximization in (33) leads to the condition

$$\frac{\mathrm{d}\mathbf{u}(K_t, K_{t+1})}{\mathrm{d}K_{t+1}} = \beta\mu \frac{\mathrm{d}\mathrm{E}_t V(K_{t+1})}{\mathrm{d}K_{t+1}}$$

or

$$u'(c_{y,t}) = \beta \frac{\mathrm{d}\mathrm{E}_t V(K_{t+1})}{\mathrm{d}K_{t+1}}. \tag{35}$$

The expression in (34) can be moved one period forward and used as the derivative of the value function in (35), which leads to

$$u'(c_{y,t}) = \beta \mathrm{E}_t[u'(c_{o,t+1})(A_{t+1}F_K(K_{t+1}, n_{t+1}) + 1 - \zeta_{t+1})].$$

A.2 Marginal utility

To see that $\hat{u}'(c) \approx \sigma\hat{c}$, use a first-order Taylor expansion to find

$$\begin{aligned}
\widehat{u'}(c) &\approx \frac{u'(c) - u'(\bar{c})}{u'(\bar{c})} \\
&\approx \frac{(c - \bar{c})u''(c)}{u'(\bar{c})} \\
&\approx \frac{\bar{c} \cdot u''(\bar{c})}{u'(\bar{c})} \cdot \hat{c} \\
&= -\sigma\hat{c}.
\end{aligned}$$